Journal of Statistical Physics, Vol. 55, Nos. 3/4, 1989

Stochastically Perturbed Landau–Ginzburg Equations

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Received August 2, 1988; revision received December 2, 1988

We analyze several aspects of a reaction-diffusion equation in two space dimensions with cubic nonlinearity, stochastically perturbed by white noise in time and in space. This equation needs renormalization, and physical implications of this circumstance are discussed. In particular, for sufficiently large coupling constant the effective potential becomes a double well and rare transitions from one minimum to the other are possible. These, however, are revealed only by large-scale fluctuations which exhibit a bimodal distribution. Fluctuations below a critical scale have unimodal distribution and do not "see" the double well. This phenomenon is connected with the singular character of local fluctuations in two or more space dimensions. The theoretical results are confirmed by numerical simulations. The possible physical relevance of our results is illustrated in connection with the analysis of certain observations of atmospheric fields.

KEY WORDS: Stochastic P.D.E.; renormalization; large (small) scale fluctuations; atmospheric bimodality.

1. INTRODUCTION

Landau–Ginzburg (LG) equations [see (2.1)] are widely used to model a variety of different physical situations. Although they were originally motivated by the study of the order parameter near the critical point in the theory of superconductivity,[(1)] their use has been extended to describe Bénard cells near the transition to the convection mode,[(2)] the effect of orbital forcing on the earth's climates,[(3)] the nonlinear optical bistability in lasers,[(4)] crystallization phenomena,[(5)] etc. Generally speaking, LG equations are phenomenological equations representing an approximate macroscopic description of the physical phenomenon under study. Their justification at a deeper level would require a microscopic analysis, which is usually a very difficult task. Therefore, what one very often does is to simulate or

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0022-4715/89/0500-0505$06.00/0

model some of the neglected effects connected with the microscopic nature of the system in terms of appropriate perturbations to the original macroscopic equation. An important class of such perturbations is represented by external noise, which according to different circumstances may describe part of the neglected complicated structure of the system or external disturbances or both. The choice of the external noise is not a trivial point. White noise is very often a candidate and the rationale behind this choice is just our lack of a sufficiently detailed knowledge of the physical situation. In fact the basic feature of a white noise is that it gives equal weight to all scales of time and space. Although this may be rather unphysical in certain cases, it seems reasonable to expect that when a white noise of small amplitude is added to our deterministic equation the effect of the small scales should not be overwhelming in determining the macroscopic behavior of the system. In other words, using a terminology borrowed from quantum field theory, our equations should exhibit a property of "ultraviolet stability." Now it is a fact that even a simple scalar LG equation does not have this property when the dimension of space is larger than 1. This poses a methodological problem. A possible reaction is to abandon our model momentarily and try to obtain a more accurate description of the physics involved. But another possible attitude (perhaps not equally satisfactory in the long term, but immediately more constructive) consists in asking the following question: is it possible to construct a "minimal" modification of our equation that exhibits the property of ultraviolet stability? This is the kind of question with which quantum field theorists have been struggling for several decades and for which the theory of renormalization in its different versions has been developed. Therefore a possible reformulation of the previous question is: is it possible to renormalize our LG equation? The answer is affirmative and in fact at the level of perturbation theory this problem was discussed long ago in connection with dynamical critical phenomena.[6] Recently in space dimension $d=2$ progress has been made by giving a completely nonperturbative treatment of the problem[7] including a study of the behavior of the equation when the noise becomes small.[8] This treatment is based on a combination of stochastic calculus with the methods of constructive field theory.

The main purpose of this paper is to discuss qualitatively the basic nonperturbative features of the model in order to clarify its possible uses in interpreting phenomenological situations. In particular, we shall analyze two aspects. First we consider the mechanism which allows a bifurcation from a one-equilibrium to a two-equilibrium regime with emphasis on the differences with respect to the deterministic case. Then we analyze the behavior of fluctuations at different scales. This is quite remarkable, as the two-equilibrium regime is revealed only by the large-scale fluctuations.

Only these scales exhibit a bimodal distribution, while below a critical scale the probability distribution of a suitable averaged field is always unimodal. This aspect, which is here examined in detail for the first time, is strictly connected with the singular character of fluctuations over small scales. We remark that the behavior we have just described is absent in one space dimension, as we discuss in the Appendix.

We then discuss computer simulations of the nonperturbative features of the model. An extensive numerical study is presented which, in our opinion, beautifully illustrates the theoretical analysis. Finally, the possible physical relevance of the above scenario is illustrated by considering an important problem in atmospheric physics.

At a more general level we would like to summarize the content of this paper by stressing the following points.

1. We start from a model, a reaction-diffusion equation, which is believed to be relevant in the study of various hydrodynamical problems.

2. We perturb it with a white noise in all variables to obtain a scheme applicable to many situations where perturbations also on scales much smaller than the macroscopic ones of interest are active. For example, in the problem of atmospheric physics considered at the end, stochastic perturbations exist down to the molecular level, i.e., the scale of the mean free path.

3. If the scheme is not stable, we introduce a *minimal* modification (renormalization) which ensures "ultraviolet" stability. In the present context this means *stability under perturbations on scales smaller than the scales that the equation is supposed to describe.*

We believe that the above points may have a general methodological significance because they actually introduce a criterion for generating acceptable perturbed hydrodynamic equations. The situation is reminiscent of quantum field theory and elementary particle physics, where models are often adopted with a view toward their renormalizability. Of course, if our attitude is correct, the renormalized stochastically perturbed equations should be obtained from the microscopic dynamics. This is a formidable challenge. However, recent studies of the hydrodynamic limit of certain microscopic models suggest that the problem we raise may be within the reach of modern techniques of nonequilibrium statistical mechanics.[14]

2. RENORMALIZATION OF LANDAU–GINZBURG EQUATION

The prototype of the LG equation is

$$\partial_t \varphi = \nu \, \Delta\varphi - m\varphi - V'(\varphi) \tag{2.1}$$

where $V' = \delta V/\delta\varphi$ is the "potential" and we shall assume hereafter that V is a polynomial in φ; ν and m are constants; and φ is a scalar function which depends on $(x, t) \in \Omega \times [0, T]$, and Ω is a domain in R^d. We are interested in studying the effect of stochastic perturbations on (2.1). More precisely, we add to the rhs a term $\varepsilon^{1/2}\, dW(x, t)/dt$, obtaining the stochastic partial differential equation

$$\partial_t \varphi = \nu\, \Delta\varphi - m\varphi - V'(\varphi) + \varepsilon^{1/2}\, dW/dt \tag{2.2}$$

The white noise is defined by

$$\begin{gathered} E\left[\frac{dW}{dt}\right] = 0 \\ E\left[\frac{dW(x, t)}{dt}\frac{dW(x', t')}{dt'}\right] = \delta(x - x')\,\delta(t - t') \end{gathered} \tag{2.3}$$

Let Z_t be a solution of the linear stochastic partial differential equation

$$\partial_t Z_t = \nu\, \Delta Z_t - mZ_t + \varepsilon^{1/2}\, dW/dt \tag{2.4}$$

Z_t is given by

$$Z_t(x) = Z_{0t}(x) + \varepsilon^{1/2} \int_0^T \int_\Omega G(x, x', t, t')\, dW_{t'}(x')\, dx', \qquad t, t' < T \tag{2.5}$$

where Z_{0t} is a solution of the homogeneous equation and $G(x, x', t, t')$ is the fundamental solution of the operator $(\partial_t - \nu\, \Delta + m)$, i.e.,

$$\begin{gathered} (\partial_t - \nu\, \Delta + m)\, G(x, x', t, t') = \delta(x - x')\,\delta(t - t') \\ G(x, x', t, t') = 0 \qquad \text{for} \quad t < t' \end{gathered} \tag{2.6}$$

Choosing appropriate boundary conditions in Ω, we can solve (2.6) by expanding G in eigenfunctions of the Laplace operator φ_k defined by

$$\Delta\varphi_k = -\lambda_k \varphi_k \tag{2.7}$$

If Ω is a square, then φ_k are Fourier modes. We easily obtain

$$G(x, y, t, s) = \theta(t - s) \sum_k e^{-(m + \nu\lambda_k)(t - s)} \varphi_k(x)\, \varphi_k(y) \tag{2.8}$$

where θ is the Heaviside function. Analogously, we can expand the noise in terms of φ_k:

$$W(x, t) = \sum_k W_k(t)\, \varphi_k(x) \tag{2.9}$$

where W_k are a sequence of independent Wiener processes. From (2.5) and (2.8) it follows that

$$\int_\Omega E(Z_t^2(x))\,dx = \int_\Omega Z_{0t}^2(x)\,dx + \frac{\varepsilon}{2}\sum_k \frac{1}{m+\nu\lambda_k}\,[1 - e^{-2t(m+\nu\lambda_k)}] \tag{2.10}$$

This series diverges for $d \geqslant 2$ and the divergence is logarithmic for $d = 2$. Let us now write (2.2) as an integral equation using the Gaussian process Z_t:

$$\varphi_t(x) = Z_t(x) - \int_0^T \int_\Omega G(x, x', t, t')\, V'(\varphi_{t'}(x'))\,dx'\,dt' \tag{2.11}$$

Due to (2.10), any nonlinear term in $V'(\varphi)$ is divergent and is not a stochastic variable. Therefore (2.11) is meaningless. In order to illustrate how to give a meaning to our equation, we specialize $V'(\varphi)$: $V'(\varphi) = g\varphi^3$, where g is a constant. We first introduce a cutoff noise by restricting the sum in (2.9) to $k \leqslant \Lambda$. We then try to modify φ^3 in a way that it remains a good stochastic variable, in the limit $\Lambda \to \infty$. A standard way to do this is to replace φ^3 with its Wick product:

$$\varphi^3 \to \varphi^3 - 3E(Z_{t=\infty}^2)\varphi \equiv :\varphi^3: \tag{2.12}$$

This procedure, inspired by experience in quantum field theory, has been rigorously shown to be appropriate for a slightly different equation.[7] It can be shown that $:\varphi^3:$ is a good stochastic variable since all its moments exist. Hence our modified equation reads

$$\varphi_t = Z_t - g\int_0^T \int_\Omega G :\varphi_{t'}^3:\,dx'\,dt' \tag{2.13}$$

or in differential form

$$\partial_t \varphi = \nu\,\Delta\varphi - m\varphi - g\varphi^3 + 3gE(Z_{t=\infty}^2)\varphi + \varepsilon^{1/2}\frac{dW}{dt} \tag{2.14}$$

In the next section we shall explain how the methods of quantum field theory allow us to give a meaning to the evolution described by (2.13).

3. RELATION WITH QUANTUM FIELD THEORY

Equations like (2.13) or (2.14) have been studied for the first time in connection with the so-called stochastic quantization of field theories. In this section we will summarize the main insights which have been obtained in such a study. One has to realize immediately that even after renormalization, an equation like (2.13) or (2.14) cannot be taken in a "strong" sense, that is, as a relationship satisfied by an appropriate process that we call its solution. This is due to the explicit infinity introduced by the Wick product. The way out is provided by the notion of weak solution in a probabilistic sense. This means the following. A stochastic process $\varphi_t(x)$ is called a weak solution of (2.13) if the process $\hat{Z}_t(x)$ defined by

$$\hat{Z}_t(x) = \varphi_t(x) + g \int_0^T \int_\Omega G(x, x', t, s) :\varphi_s(x')^3: dx'\, ds \tag{3.1}$$

has the same probability distribution as Z_t solution of (2.4). In other words, while it is not possible to find a functional $\varphi_t(Z)$ which satisfies (2.13), taking (2.13) as a definition of $\hat{Z}_t$, we can find a φ_t that gives, via (3.1), a Gaussian process in distribution identical with Z_t.

Then we describe how to construct φ_t. One starts from (2.13) with a cutoff noise $W_{t\Lambda}$. In this case (2.13) has a strong solution which defines a Markovian semigroup via the Girsanov formula[7]:

$$E_{\varphi_{0\Lambda}} f(\varphi_{t\Lambda}) = E_{\varphi_{0\Lambda}} f(Z_{t\Lambda} e^{\eta_{t\Lambda}}) \tag{3.2}$$

where

$$\eta_{t\Lambda} = -\frac{g}{\varepsilon^{1/2}} \int_0^t (:Z_{s\Lambda}^3:, dW_{s\Lambda}) - \frac{g^2}{2\varepsilon} \int_0^t ds\, \| :Z_{s\Lambda}^3: \|^2 \tag{3.3}$$

where (..., ...) is the scalar product corresponding to the space integration, $\|...\|$ is the induced norm, and f is any bounded functional of $\varphi_{t\Lambda}$. Using the methods of constructive field theory, it was shown in ref. 7 that for the case of an equation slightly different from (2.13) it is meaningful to take the limit $\Lambda \to \infty$ in (3.2). Since the choice of the equation discussed in ref. 7 was dictated mainly by technical simplicity, we shall assume that the conclusions extend also to (2.13). That is, we shall assume that φ_t can be defined for $\Lambda \to \infty$ by

$$E_{\varphi_0} f(\varphi_t) = E_{\varphi_0} f(Z_t e^{\eta_t}) \tag{3.4}$$

The definition of the weak solution implies that expectation values calculated using $\varphi_{t\Lambda}$, for large Λ, approximate well the expectation values calculated with φ_t.

In ref. 7 it was also proved that φ_t is ergodic and mixing and that its stationary measure is the usual Euclidean quantum $:\varphi_2^4:$ model.

The Wick product changes the original single minimum potential into a double-well potential. Consequently, the equilibrium measure of (2.13) in the infinite-volume limit exhibits a phase transition when g reaches a critical value.[9] To study whether there is a trace of this phenomenon in our finite-volume equations (2.13) or (2.14), let us change the Wick product (2.12) through the following formal manipulation:

$$\begin{aligned} m\varphi + g\varphi^3 - 3gE(Z_{t=\infty}^2)\varphi &= m\varphi + g\varphi^3 - 3gE(Z_{t=\infty}^{\alpha 2})\varphi \\ &\quad - 3g[E(Z_{t=\infty}^2) - E(Z_{t=\infty}^{\alpha 2})]\varphi \end{aligned} \tag{3.5}$$

where Z_t^α is the process whose variance is given by

$$\int_\Omega E(Z_t^{\alpha 2}(x))\,dx = \frac{\varepsilon}{2}\sum_k \frac{1}{\alpha + \nu\lambda_k}\left[1 - e^{-2t(\alpha + \nu\lambda_k)}\right]$$

If $\alpha > m$, the coefficient of the last term is obviously negative and for sufficiently large g can dominate over the mass term m. In this case everything goes as if we had started from a potential

$$V(\varphi) = \tfrac{1}{4}g\varphi^4 - \tfrac{1}{2}\rho\varphi^2 \tag{3.6}$$

with

$$\rho = 3g[E(Z_{t=\infty}^2) - E(Z_{t=\infty}^{\alpha 2})] - m \tag{3.7}$$

Notice that ρ is independent of Λ for $\Lambda \to \infty$, as follows from a very simple calculation. This is a double-well potential with two minima $\pm\varphi_c$ given by

$$\varphi_c = (\rho/g)^{1/2} \tag{3.8}$$

Near each minimum we may define a mass equal to

$$V''(\varphi_c) = 2\rho \tag{3.9}$$

If we require now

$$2\rho = \alpha \tag{3.10}$$

which is a transcendental equation for α, we may intepret the term $-3gE(Z_{t=\infty}^{\alpha 2})$ appearing in (3.5) as a renormalization of fluctuations near each of the new minima. Therefore, if g is large enough so that (3.10) has a solution $\alpha \gg m$, the Wick product becomes relevant and produces an effective double-well potential of finite depth. It is interesting to remark that the bifurcation at $m = 0$ of the deterministic equation now can occur at $m > 0$. In finite volume, jumps are possible between the two minima and should appear in the solutions of our stochastically perturbed LG equation.

4. FLUCTUATIONS

In this section we discuss in some detail the nature of fluctuations of the field when the noise intensity goes to zero. We first show that there is a regime in which the renormalization term is irrelevant. Qualitatively this phenomenon can be understood in the following way. If we consider the cutoff equation, this has strong solutions and one can apply to it the usual theory of small, random perturbations. For ε (the noise intensity) sufficiently small the renormalization term is of the order of $\varepsilon \ln \Lambda$ and can be consistently disregarded. The probability that a trajectory of the cutoff field be close to a preassigned function $f_t(x)$ in the same cutoff space is of the order of

$$P_\Lambda(\sup_{x,t} |\varphi_t - f_t| < \delta) \cong \exp\left(-\frac{I_{T\Lambda}(f_t)}{\varepsilon}\right) \tag{4.1}$$

with

$$I_{T\Lambda}(f) = \frac{1}{2}\int_0^T \|(\partial_t f - \nu\, \Delta f + mf + gf^3) P_\Lambda\|^2 \tag{4.2}$$

where P_Λ is the projection onto the subspace $k \leqslant \Lambda$. Therefore it is exponentially small unless f_t coincides with a solution of the deterministic part of (2.13) projected onto the subspace $k \leqslant \Lambda$. It is clear, however, that due to the term $\varepsilon \ln \Lambda$ the $\varepsilon \to 0$ limit and the $\Lambda \to \infty$ limit cannot be interchanged. To reach conclusions about the field φ_t after removal of the cutoff, we have to analyze the problem further. First of all we observe that the trajectories of the field φ_t live in a space of distributions insofar as the x dependence is concerned. The time dependence is continuous. This means that it is meaningful to speak of trajectories only for quantities smeared in space, e.g.,

$$\varphi_g(x) = \int g(x - x')\, \varphi_t(x')\, dx' \tag{4.3}$$

where g is an appropriate test function. If R_0 is the diameter of the region over which $g(x) \neq 0$, (4.3) is a field with an effective cutoff of the order of R_0^{-1}. To simplify the discussion, we then analyze in the $\varepsilon \to 0$ limit the behavior of φ_t^n, i.e., the projection of our distribution-valued process on the first $n \cong R_0^{-1}$ vectors of the orthonormal basis. φ_t^n is a process continuous in all variables t and x. One can now expect that the probability of large deviations for φ_t^n must be close to the probability of large deviations for $\varphi_{t\Lambda}^n$, i.e., the projection of the process solution of the equation with cutoff noise on the first n eigenvectors of the basis, at least if $n \ll \Lambda$. That this type of guess is correct has been proven recently[8] for an equation similar to (2.13) and we assume again that the result can be extended to our case. The conclusion of this part of the discussion is therefore that provided we average our field φ_t over a finite scale, it will tend to be close when $\varepsilon \to 0$ to the average of the solutions of an unrenormalized deterministic equation. The dependence of the probability of large deviations on the scale considered is contained in the formula

$$P(\sup_{x,t} |\varphi_t - f_t| < \delta) \cong \exp\left(-\frac{I_*(f)}{\varepsilon}\right) \tag{4.4}$$

where

$$I_*(f) = \operatorname*{Inf}_{\bar{f}:\bar{f}^n = f} I_{T\Lambda}(\bar{f}) \tag{4.5}$$

where $I_{T\Lambda}$ is given by (4.2) and $\bar{f}^n$ is the projection of $\bar{f}$ on the first n vectors of the basis.

In the previous discussion the coupling constant g has been kept fixed while $\varepsilon \to 0$. In real life, however, ε is finite and $g\varepsilon$ may not be too small that equation (3.10) of the previous section has a nontrivial solution. In this situation the previous theory of large deviations has to be supplemented by additional considerations. Suppose in fact that we keep $g\varepsilon \gg 1$ and constant so that also α is constant when ε vanishes. Then it is clear that we cannot take the limit $\varepsilon \to 0$, as the drift explodes. Furthermore, both the location of the wells (3.8) and their depth vanish with ε. Therefore, if we want to see the double well, we have to keep ε small but finite. In spite of this, we believe that qualitatively the main features of the usual analysis of a process in a double well based on large-deviation theory applies to our case as well. Thus, we expect the equilibrium distribution to be bimodal. However, there are some aspects of our situation which have to be taken into account. First of all, as we already remarked, because of the distribution character of the field realizations, fluctuations depend strongly on the scale considered. In fact, fluctuations on a scale R_0 around one of the minima of the potential are easily seen to be of the order of

$$\varepsilon \ln \frac{(R_0^{-1})^2}{\alpha} \tag{4.6}$$

On the other hand, φ_c^2 is of the order

$$\varepsilon \ln \frac{\alpha}{m} \tag{4.7}$$

Therefore, when

$$\frac{(R_0^{-1})^2}{\alpha} > \frac{\alpha}{m} \tag{4.8}$$

we expect the bimodal structure of the equilibrium probability density distribution to be strongly modified and eventually canceled. Thus, bimodality will be a feature of large-scale fluctuations. It is interesting to remark that it is enough to smear the field over large distances with respect to only one coordinate to obtain a weakly fluctuating and therefore bimodal quantity.

5. NUMERICAL EXPERIMENTS

In this section we present some numerical simulations in order to illustrate the previous theoretical discussion.

To simulate numerically (2.2), we use finite differences on a regular lattice of spacing a. Each node is identified by the indices $j, k = 1,..., N$ and we chose $aN = 1$. Then (2.2) becomes

$$d\varphi_{j,k} = \left(\frac{v}{a^2} D\varphi_{j,k} - m\varphi_{j,k} - g\varphi_{j,k}^3\right) dt + \frac{\varepsilon^{1/2}}{a} dW_{j,k}(t) \tag{5.1}$$

where

$$D\varphi_{j,k} = \varphi_{j+1,k} + \varphi_{j,k+1} + \varphi_{j-1,k} + \varphi_{j,k-1} - 4\varphi_{j,k}$$

In (5.1), $dW_{j,k}/dt$ is a white noise and

$$E\left[\frac{dW_{j,k}(t)}{dt}\frac{dW_{l,q}(t')}{dt'}\right] = \delta_{j,l}\,\delta_{k,q}\,\delta(t-t')$$

where $E[\ldots]$ stands as usual for averaging over the noise realizations. In all numerical simulations we used periodic boundary conditions and $v = 0.1$, $\varepsilon = 0.01$, and the time step Δt small enough to ensure numerical stability, i.e., $\varepsilon\,\Delta t \ll a^2$. The computational cost for $N = 16$ is about 0.0027 sec for time step on a IBM 3090/VF.

We first consider the linear case of Eq. (5.1):

$$dZ_{j,k} = \left(\frac{v}{a^2} DZ_{j,k} - mZ_{j,k} - \right) dt + \frac{\varepsilon^{1/2}}{a} dW_{j,k}(t) \tag{5.2}$$

Let $G(m, a)$ be the mean variance of $Z_{j,k}$, i.e., the quantity $(1/N^2) \sum_{j,k} Z_{j,k}^2$ at equilibrium. It is quite simple to show that

$$G(m, a) = \sum_{p,q} \frac{\varepsilon a^2}{2S(m, v, q, p)} \tag{5.3}$$

Table I. The Behavior of the Roots of the Transcendental Equation (5.5) As a Function of g

g	130	150	200
α_1	1.93	4.00	8.54
α_2	1.07	0.57	0.33

where

$$S(m, v, q, p) = ma^2 + 4v - 2v\cos(2\pi q/N) - 2v\cos(2\pi p/N)$$

and $1 \leqslant q,\ p \leqslant N$. Here $G(m, a)$ is used to obtain the renormalized version of (5.1)

$$d\varphi_{j,k} = \left(\frac{v}{a^2} D\varphi_{j,k} + \mu\varphi_{j,k} - g\varphi_{j,k}^3\right) dt + \frac{\varepsilon^{1/2}}{a}\, dW_{j,k}(t) \tag{5.4}$$

where $\mu = -m + 3gG(m, a)$. The discretized version of (3.10) is

$$\alpha = -2m + 6g[G(m, a) - G(\alpha, a)] \tag{5.5}$$

As in the continuum case, (5.5) is a transcendental equation for the α. We have solved (5.5) numerically for different values of a and g by employing a Newton method. It turns out that for $a \leqslant 1/16$ and $g \geqslant 130$, (5.5) has two solutions α_1 and α_2. For larger values of g, $\alpha_1 \gg \alpha_2$ (see Table I). We have also studied the a dependence of α_1 at $g = 140$, as shown in Table II. For sufficiently small value of a, α_1 tends to be constant, as expected.

We want to study the statistical properties of $\varphi_{j,k}$ for different values of g keeping a constant. In particular, we will present some numerical results for $a = 1/16$. One may argue that this value of a is not small enough to guarantee that the statistical properties of $\varphi_{j,k}$ are independent of a (see again Table II). However, according to Sections 2 and 3, once we use the renormalized equation, it can be expected that the a dependence of all the relevant properties is rather weak.

Table II. The Behavior of the Largest Root of (5.5) and the Value of the New Equilibrium after Renormalization As a Function of $N = 1/a$

N	16	32	64	128	256
α_1	3.06	2.45	2.23	2.15	2.13
φ_c	0.1046	0.0933	0.0891	0.0876	0.0869

To discuss the scale dependence of the statistical properties of $\varphi_{j,k}$, we consider the quantities

$$\psi(M)=\frac{1}{M^2}\sum_{j,k=1,\ldots,M}\varphi_{j,k} \tag{5.6}$$

For $M=N$, $\psi(M)$ corresponds to the spatial average of $\varphi_{j,k}$, while for $M=1$, $\psi(1)$ is $\varphi_{1,1}$, i.e., the value of the field φ in one point. Due to the periodic boundary conditions it is not important where the square of side M is centered. We will denote by $P(M, g)$ the stationary probability distribution of $\psi(M)$ for a given value of g and $a=1/16$. For g sufficiently large, namely for $g\geqslant 130$ according to Table I, $P(16, g)$ should be bimodal with maxima located near $\varphi_c=\pm(\alpha_1/g)^{1/2}$. On the other hand, for a small value of g, $P(16, g)$ should be unimodal. We can deduce $P(M, g)$ starting from the time record of $\psi(M)$ obtained by direct numerical integration of (5.4). Deduction of the probability density distribution (pdd) from the data is a problem in itself and deserves some discussion. Let $\psi_i(M)$ be the i iterate of $\psi(M)$, i.e., $\psi(M)$ at time $i\,\Delta t$. Because of obvious limitations, we will have a finite, say T, sequence of $\psi_i(M)$. We want to estimate the pdd of $\psi_i(M)$, $i=1, 2,\ldots, T$. This problem has a long tradition and recent developments in the theory of nonparametric density estimation suggest the use of a generalization of the concept of histogram as a potential estimator. In this paper we employ a quartic kernel estimator, i.e., an estimate of the true pdd at the datum $\psi_i(M)$ given by

$$P[\psi_j(M)]=N_P\sum_{n=1}^{T}\sum_{j=1}^{Q}\frac{15}{16}\left[1-\left(\frac{\psi_j(M)-\psi_n(M)}{h}\right)^2\right]^2 \tag{5.7}$$

where N_P is a normalization constant, h is the mesh interval centered around $\psi_j(M)$ over which we estimate the probability density distribution, and Q is the total number of points for which we estimate the pdd. In virtue of (5.7), the estimate depends on the window size h and its choice is not suggested by the data. To establish the optimal choice of the window, we can proceed as follows. We note that for large h the estimate obtained by using (5.7) is always a unimodal pdd, while for h very small the estimate will be always a multimodal pdd. Thus, starting from large h, we can decrease the window size to the value, say $h=h_c$, for which an additional mode appears. Now the problem is turned into establishing the statistical significance of this bimodal pdd. For this purpose we can proceed as follows. Because the statistical significance will depend on the number of independent observations, we must give a criterion to decide how many independent observations are contained in our sample $\{\psi_i(M)\}$. In this paper we use the simple rule of taking as independent observations a datum every T_d, where T_d is the decorrelation time of our solution.

Let N_{ind} be the number of independent observations contained in our record; we can prepare synthetic data sets, each of length N_{ind}, by sampling a unimodal pdd, for example, the pdd of our original data set obtained with a window size slightly greater than the above-defined h_c. Then, we estimate the pdd's of each synthetic data set by using a window size $h = h_c$ and determine the percentage of cases which a bimodal pdd occurs. This percentage will give us an estimate of the probability that sampling N_{ind} observations from a unimodal pdd, our estimated pdd has a bimodal structure due to the arbitrariness of the window size. If this probability is large, then we reject our hypothesis of bimodality. If it is small, we reject the null hypothesis that the true distribution is unimodal. So we have established a confidence level for the modal structure of the true pdd.

We first consider the case of g small, namely $g = 10$. We recall that small and large values of g means that (5.5) has no solution or two solutions.

We estimate $P(16, 10)$ using (5.7) with $T = 10{,}000$, $T_d = 10$, and $Q = 41$. For $h = 0.09$ a bimodal pdd is obtained. In accordance with the above discussion, the statistical significance is assessed by constructing 100 sample sets, each of length $N_{\text{ind}} = 1000$, by sampling from a unimodal pdd. The unimodal pdd is chosen to be the estimated pdd of the data $\{\psi_i(M)\}$ which is obtained by setting $h = 0.12$ in (5.7). We estimated the pdd's for each of these 100 sample sets using $h = 0.09$ and we found that in 50% of the cases the estimates were bimodal pdd's. Hence, we conclude that the

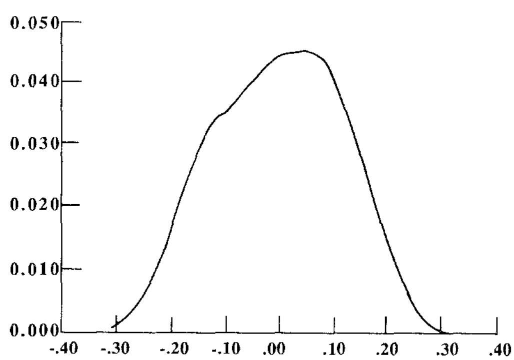


Fig. 1. Probability density distribution $P(16, g)$ for the numerical simulation of (5.1) at "small" $g = 10$. The probability density distributions plotted in this and the following figures are not normalized.

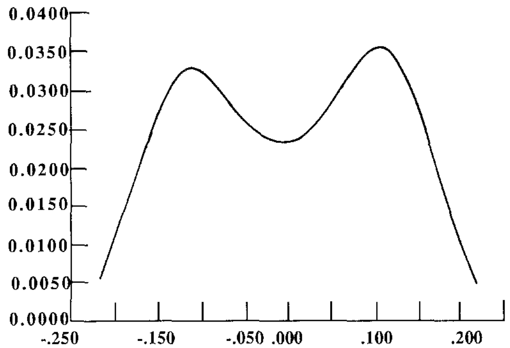


Fig. 2. Probability density distribution $P(16, g)$ for the numerical simulation of (5.1) at "large" $g = 140$.

appearance of the new mode is artificially introduced by the smallness of the window size and we accept the hypothesis that the true pdd has unimodal structure.

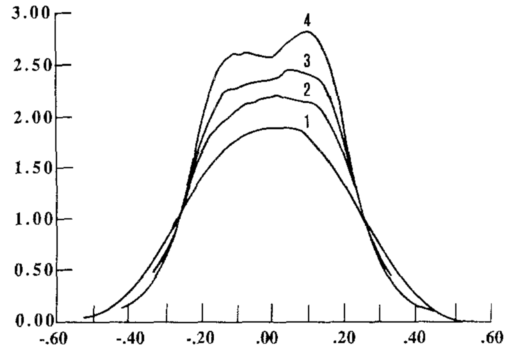
The same procedure has been used to estimate $P(16, 140)$. In this case at $h = 0.12$ a bimodal pdd is observed and the estimate is significant at 96%. Figure 1 and 2 show $P(16, 10)$ and $P(16, 140)$, respectively. The numerical solution of (5.5) for $g = 140$ is $\alpha_1 = 3.066$, which corresponds to $\varphi_c = \pm 0.104$. This value compares well with the maximum of $P(16, 140)$.

As previously mentioned, we are interested in studying $P(M, 140)$ as a function of M. We already know that for decreasing values of M the variance of the field $\psi(M)$ increases. For small values of M the fluctuations are large enough to mask the double-well potential. In this case $P(M, 140)$ becomes unimodal.

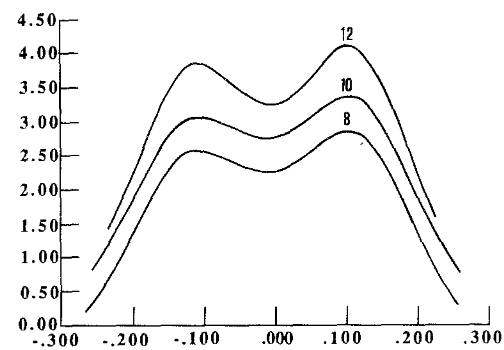
In Fig. 3 we show $P(M, 140)$, $M = 1, 2, 3, 4, 8, 10, 12$, obtained by the numerical integration and using the same statistical techniques already described. We see that for $M < 4$ the probability distribution becomes unimodal.

6. A POSSIBLE APPLICATION TO ATMOSPHERIC PHYSICS

A remarkable feature of the present model is the absence of a bimodal distribution at very small scale. This property may have relevance for some geophysical problems, among which we report a case which we have recently discussed in other publications.[10–12] In these papers the aim was to establish whether the atmosphere possesses more than a weather regime



(a)



(b)

Fig. 3. Probability density distributions $P(M, 140)$ for the numerical simulation of (5.1) for (a) $M = 1, 2, 3, 4$, (b) $M = 8, 10, 12$. The numerical labels on the curves refer to the different values of M.

in accordance with theories of large-scale motion.[13] To this purpose in ref. 10 it was proposed to investigate the following indicator. Let $Z(\lambda, \theta, t)$ be the 500-mb geopotential height during the Northern Hemisphere winter (artificially defined as the period from the 1 December to the 28 March), where λ, θ are respectively the latitude and the longitude of a point and t the time. This field was considered as a stochastic variable. We define the latitudinal averaged field as

$$\bar{Z}(\theta, t) = \int_{\lambda_1}^{\lambda_2} Z(\lambda, \theta, t)\, d\lambda$$

where $[\lambda_1, \lambda_2]$ is a relatively small interval with respect to the planetary scale. Let us consider the Fourier transform along the θ direction:

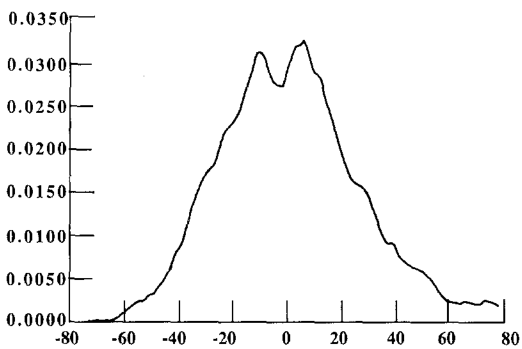
$$\bar{Z}(\theta, t) = \sum_k A(t)_k \, e^{ik\theta}$$

The interesting indicators are

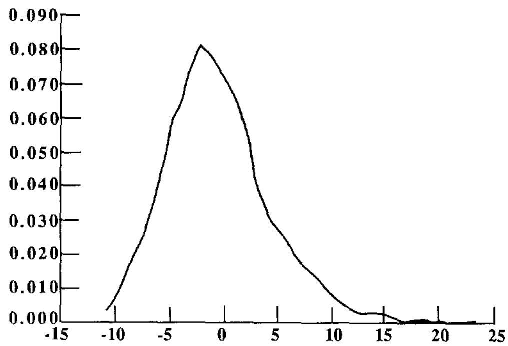
$$\Gamma_L(t) = \left[\sum_{k=2}^{4} A_k(t)^2 \right]^{1/2} \tag{6.1}$$

and

$$\Gamma_S(t) = \left[\sum_{k=7}^{18} A_k(t)^2 \right]^{1/2} \tag{6.2}$$



(a)



(b)

Fig. 4. Probability density distribution of (a) Γ_L [defined by (6.1)] and (b) Γ_S [defined by (6.2)] for the atmospheric data discussed in the text.

These two variables represent the planetary-wave (large-scale) and the short-wave (small-scale) behavior of Z. Alternatively, Γ_L and Γ_S can be considered as the large-scale and the small-scale averages of the two-dimensional stochastic field $Z(\lambda, \theta, t)$. In Figs. 4a and 4b we show the probability density estimates of Γ_L and Γ_S induced by a set of data for ten winters. It appears that Γ_L possesses a bimodal density distribution, while Γ_S does not. Now, if the fitting processes of $Z(\lambda, \theta, t)$ (i.e., a stochastic variable having the same density distribution of Γ_L and Γ_S) is controlled by an equation similar to the Landau–Ginzburg stochastic equation, arguments such as the one proposed in the previous sections hold and the peculiar dependence of the bimodality on the averaging scale may have its explanation in the renormalization process. Of course, the observed behavior of Γ_L and Γ_S may have a different explanation; we feel, however, that the mechanism discussed in this paper provides a very illuminating guideline.

APPENDIX. THE LG EQUATION IN ONE DIMENSION

In one dimension renormalization of the LG equation is not necessary because $E(Z_t^2(x)) < \infty$ for all t. In order to emphasize the difference between one and two dimensions, let us consider also in $d=1$ the renormalized equation

$$\partial_t \varphi = \nu\, \partial_x^2 \varphi - m\varphi - g :\varphi^3: + \varepsilon^{1/2}\, dW/dt \tag{A.1}$$

where $:\varphi^3:$ is defined in (2.12). For $\varepsilon = 0.01$, $\nu = 0.01$, $g = 140$, and $m = 0.1$ the *bare* potential of (A.1) is a double well with two minima at ± 0.1. However, in this case we do not expect, due to the finiteness of local fluctuations, that the *effective* potential will become a double well. This is confirmed by numerical integration of (A.1); see Fig. 5.

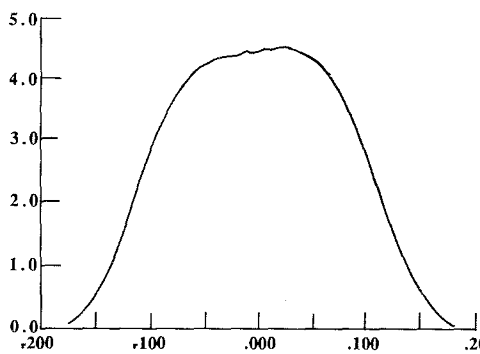


Fig. 5. Probability density distribution obtained by numerically integrating (A.1) and averaging on the whole one-dimensional domain. The *bare* potential is a double well with two minima at ± 0.1.

ACKNOWLEDGMENTS

We thank the IBM European Center for Scientific and Engineering Computing in Rome, where all the computations have been performed using IBM 3090 vector multiprocessors. A.S. was partially supported by NSF grant 8507496 and by INFN Sezione di Roma.

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